different from zero. Such coefficients were eliminated and others tried. Only those significantly different from zero were finally kept.

### III. RESULTS

The values of the goodness of fit parameter $\chi^2$ is shown in Table I for the Scotti-Wong[4] model published phases, and for the Yale[5] energy-dependent phase analyses YLAM and YRB1. The values for two comparison representations are also listed: The corresponding polynomial coefficients are shown in Table II.

Examination of the contributions to $\chi^2$ from the individual data points revealed that the single datum (10.2°) at 98 MeV contributed 421 to $\chi^2$ for CR(21). The five lowest-angle cross section points at 98 MeV (including the 10.2° point) contributed a total of 548. It is to be strongly recommended that these data and their associated experimental standard deviations be re-examined.

It would appear that the Scotti-Wong model, as represented by the published phases, is rather poor if judged as a phenomenological model against CR(10). Quite different criteria should be applied, of course, if the Scotti-Wong model is judged theoretically.

The computations reported here were carried out in the Computation Center of the Pennsylvania State University and the Atomic Energy Commission Computation Center at New York University.

[4] A. Scotti and D. Y. Wong, Phys. Rev. Letters **10**, 142 (1963).
[5] G. Breit, M. H. Hull, Jr., K. E. Lassila, and K. D. Pyatt, Jr., Phys. Rev. **120**, 2227 (1960).

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# Negative Pion Capture From Rest on Complex Nuclei

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The capture of negative pions from rest by light nuclei, primarily $O^{16}$, is investigated by means of a shell-model calculation. The conventional pion-nucleon $ps$–($ps$) interaction is used to calculate the probability of one-nucleon and two-nucleon ejection from the nucleus. Due to the effects of distorted waves, one-nucleon ejection is not found to be suppressed as has been previously supposed but is comparable to the two-nucleon mode. It is also found that back-to-back ejection of a nucleon pair is dominant over parallel ejection, and that the capture probability as a function of the angle between an ejected pair will show structure.

## I. INTRODUCTION

THOUGH the capture of the $\pi^-$ meson from rest by deuterons was used 13 years ago[1,2] to determine that the pion was a pseudoscalar, little experimental or theoretical work since has developed on the capture from rest by more complex nuclei. More recently, interest has developed experimentally[3–5] on such processes and for the first time experiments are being done using counters[5] rather than emulsion or cloud chamber techniques. Experimental data are sparse and for the most part, with a few exceptions,[6,7] theoretical calculations of the various modes of capture[8] are nonexistent.

In the early theoretical work of Brueckner, Serber, and Watson[2,9] calculations were made of $\pi^-$ capture on complex nuclei by means of extrapolating the deuterium capture in an obvious way by saying that

$$(1/Z)\sigma[\pi^-+A\rightarrow \text{star}]=\Gamma\sigma[\pi^-+D\rightarrow 2n], \quad (1)$$

where the left-hand side of this expression contains the cross section $\sigma$ for absorption on a nucleus of number $A$ and the right-hand side contains a factor $\Gamma$ which allows in a vague way for the effect of the remaining $(A-2)$ nucleons on the capturing pair. Since no analytic expression for the pion-nucleon interaction existed at the time of their work, it was necessary to calculate $\sigma[\pi^-+A\rightarrow\text{star}]$ by means of a phenomenological $R$-matrix approach with a partial closure approximation.

This calculation suffered the additional disadvantage

[1] W. Panofsky, R. L. Aamodt, and J. Hadley, Phys. Rev. **81**, 565 (1951).
[2] K. Brueckner, R. Serber, and K. Watson, Phys. Rev. **81**, 575 (1951).
[3] P. Ammiraju and L. D. Lederman, Nuovo Cimento **4**, 281 (1956).
[4] M. Schiff, R. H. Hildebrand, and C. Giese, Phys. Rev. **122**, 265 (1961).
[5] S. Ozaki, R. Weinstein, G. Glass, E. Loh, L. Neimalu, and A. Wattenberg, Phys. Rev. Letters **4**, 533 (1960).
[6] S. G. Eckstein, Phys. Rev. **129**, 413 (1963).
[7] P. Ammiraju and S. N. Biswas, Nuovo Cimento **17**, 726 (1960).
[8] By $\pi^-$ capture we shall always mean capture from rest unless specifically stated otherwise.
[9] K. Brueckner, R. Serber, and K. Watson, Phys. Rev. **84**, 258 (1951).

that plane-wave final states were used for the ejected nucleons. As we shall see later, this can be a very poor approximation. Also, since the pion-nucleon interaction is strongly spin and momentum-dependent which Brueckner *et al.* did not allow for, the value of their calculation is limited. As they considered only the case of two ejected nucleons, their work leaves other questions unanswered about competing modes.

In a recent work, Eckstein[6] has calculated various possible branching ratios for the modes of $He^4$ capture of $\pi^-$'s. Though he finds agreement with one of two experiments done on this nucleus, his calculation is unsatisfactory from several points of view. He puts a delta-function potential into the pion-nucleon interaction in an *ad hoc* way and also uses plane-wave final states. His interaction is not a simple sum of one-body operators, but a much more complicated one. Though the answers obtained in his paper are reasonable, it is not possible to see the physical meaning of his assumptions. As no absolute transition rates for these processes have been measured, it is not possible to check Eckstein's numbers directly, only the ratios of any two.

It has recently been suggested by Erikson[10] on the basis of simple arguments, which we discuss later, that the ejection of a single neutron by means of $\pi^-$ capture is highly unlikely and that two-nucleon ejection will be a strongly preferred process. In our paper, we will show that most probably this conclusion is false. Moreover, we will elucidate other gross features of capture on complex nuclei.

For calculational purposes we will treat the case of $O^{16}$ since its doubly closed shell structure is particularly amenable to theoretical investigation. Experimentally, it is feasible to work with this nucleus as well. It will be obviously clear that the features we will illuminate are not dependent on the particular structure of $O^{16}$ but should hold in general for nuclei in this region. There are of course other features which will vary from nucleus to nucleus.

In what follows we will be concerned primarily with the calculation and comparison of one- and two-nucleon ejection—to be defined more clearly later—but a word is in order about the experimental evidence for such processes.

Much of the earlier cloud chamber work dealt with capture in flight at pion energies ranging from 15 to 300 MeV. Though we might expect $\pi^-$ and $\pi^+$ to be involved in similar capture processes,[11] it has been observed[12] that the capture cross section increases with energy in the low-energy range, and it is thus probable that it is unfair to extrapolate these experiments to capture at rest. Particularly, since 15 MeV is of the order of the separation energy of one nucleon from a nucleus we must be careful as the whole mechanism of capture could be different at rest than at low energies. Also the momentum of a 15-MeV pion is very different from the momentum of a pion at rest. In any event, pair ejection is found to be quite common in these cases.

In a $\pi^-$ capture experiment done by Azimov *et al.*,[13] neutrons were not detected and therefore it is impossible to judge how often we get one neutron ejected compared to a pair ejected. In two experiments, one by Ammiraju and Lederman[3] and the other by Schiff *et al.*,[4] a strong disagreement is found about the comparative rate of the process $He^4+\pi^- \rightarrow H^3+n$. Ammiraju gives a value of approximately 1% of all captures, while Schiff finds 30% which Eckstein's calculations tend to support. Ammiraju and his co-workers argue in their paper for a predominance of a two-nucleon ejection though they (as well as Schiff) find many cases of ejection of more than two particles, referred to in the literature as multipronged stars.

It is very important to distinguish here whether the fragments that fly out after $\pi^-$ capture are due to simultaneous ejection of many particles and fragments due to collisions by the absorbing proton, or whether one or two nucleons are ejected and *then* the highly excited daughter nucleus "boils off" these fragments and descends to some stable state.

Ammiraju *et al.* attempt to answer this question on the basis of angular correlations of the many outgoing particles, and they feel that many of the multiple processes are *not* the result of "boil-off." But their statistics are poor, their correlation measurements not precise, and their experiment is, in part, in complete disagreement with both theory and another experiment.

It is obviously true if one or two nucleons are removed from, say, the $s$ shell of $O^{16}$, then the remaining nucleus will be highly excited and will decay either by gammas or by "boil-off" to something stable. Thus, the experimental measurements of multiparticle ejections are completely compatable with the one- or two-nucleon ejection process. Multibreakup, directly as a result of $\pi^-$ capture, is physically different from "boil-off" and can be distinguished not only by the correlations as discussed by Ammiraju, but also by the *absence* of one or two nucleons of specified energies (always present if the process is a "boil-off"). Clearly much experimental work must be done here.

## II. ONE-NUCLEON EJECTION

The terms "one-nucleon capture model" and "two-nucleon capture model" have been used loosely in the literature. Sometimes the latter term is taken to mean the ejection of two nucleons from a nucleus due to the capture of a $\pi^-$. At other times it is taken to mean that the two nucleons do the capturing of the pion though

[10] T. Erikson (private communication).

[11] This is not strictly true since single-particle energies and pair correlations depend on whether the particles involved are protons or neutrons.

[12] F. H. Tenney and J. Tinlot, Phys. Rev. **92**, 974 (1953).

[13] S. A. Azimov, U. G. Guliamov, E. A. Zamchalova, M. Nizametdinova, M. Podgoretski, and A. Iuldashev, Zh. Eksperim. i Teor. Fiz. **31**, 756 (1956) [English transl.: Soviet Phys.—JETP **4**, 632 (1957)].

what this means is not well defined. The second meaning is often linked to the first in the assumption that two nucleon ejections can only be achieved by means of two-nucleon capture which is, of course, untrue. We will use an interaction that can be written as a sum of single-particle (nucleon) operators (in contrast to the operator of Eckstein), and with this operator we will calculate one-nucleon ejection and two-nucleon ejection, meaning one and two nucleons leaving in unbound states from a daughter nucleus.

The matrix element for one nucleon ejection is given by $M=\langle\psi_f|H_I|\psi_i\rangle$ where $\psi_i$ and $\psi_f$ are the initial and final states and $H_I$ is the interaction given in Eq. (7). We can easily find that the number of transitions per second for one ejected particle between the energies $W$ and $W+dW$ and with angles between $\Omega$ and $\Omega+d\Omega$ is

$$N'(W,\Omega)dWd\Omega=(2\pi/\hbar^2)\rho(k)p(\Delta)d\Delta|M|^2$$
$$N(W)dW=[mL^3k/(2\pi)^2\hbar^3]p(\Delta)d\Delta\int d\Omega|M|^2, \tag{2}$$

where $k$ is the momentum of the outgoing nucleon with $\rho(k)$ its density of final states, $L^3$ is a normalization volume which will disappear in our final result, and $p(\Delta)$ is the excitation spectrum of the final nucleus where $\Delta$ is the energy of that nucleus above the capturing state of the initial nucleus. For example, in the case that a one-hole state is a sharp level in the resultant nucleus we would have $p(\Delta)=\delta(\Delta-E)$, where $E$ is the experimentally determined energy of that level above the initial nuclear state; i.e., $E$ is a separation energy of one nucleon. We have from the conservation of energy that[14]

$$\mu c^2=\Delta+\hbar^2k^2/2m=139.2 \text{ MeV}$$

if we neglect the recoil of the final nucleus which is not serious for $A=16$.

To evaluate

$$M=\langle\psi_f|H_I|\psi_i\rangle \tag{3}$$

we take $\psi_f$ as a plane wave for any ejected nucleon times a nuclear wave function for a first approximation. For $\psi_i$ we will take a one-pion state, and we make the further approximation that the wave function can be written as the product of the wave function for $N$ nucleons times a $1s$ atomic Bohr orbit wave function for the pion.[15]

Thus we take

$$\psi_i=\psi(N)\varphi_\pi(\mathbf{r}), \tag{4}$$

where $\mathbf{r}$ is the pion coordinate. This is a fundamental step in the calculation as we have neglected the effect of $H_I$ on both $\psi(N)$ where it could radically alter the nuclear state, and on $\varphi_\pi(\mathbf{r})$. Brueckner[16] has investigated the effect of $H_I$ on the one-pion Coulomb wave function and shown that, for $Z=8$, the level shift and level broadening of this state are not serious.

For our problem the interaction Hamiltonian density is given by

$$\mathcal{H}_I=iG\psi^\dagger\gamma_5\psi\boldsymbol{\varphi}\cdot\mathbf{T}. \tag{5}$$

As is well known,[17] we may re-express this term in a nonrelativistic form

$$\mathcal{H}_I\approx\frac{iG}{2mc}\psi^\dagger\boldsymbol{\sigma}\psi\cdot\mathbf{p}_\pi\boldsymbol{\varphi}\cdot\mathbf{T}, \tag{6}$$

with $\boldsymbol{\sigma}$ given by the Pauli matrices and $\mathbf{p}_\pi$ is the pion momentum operator. As (6) does not preserve Galilean invariance we must add a term to get

$$\mathcal{H}_I\approx\frac{iG\mu}{2mc}\psi^\dagger\boldsymbol{\sigma}\psi\cdot(\mathbf{v}_\pi-\mathbf{v}_N)\boldsymbol{\varphi}\cdot\mathbf{T}. \tag{7}$$

Thus (3) becomes

$$M=L^{-3/2}\left\langle\psi(N-1)e^{i\mathbf{k}\cdot\mathbf{r}}\left|\frac{iG\mu}{2mc}\sum_j\boldsymbol{\sigma}_j\cdot(\mathbf{v}_\pi-\mathbf{v}_j)\tau_j^-\right|\times\psi(N)\varphi_\pi(\mathbf{r})\right\rangle, \tag{8}$$

where we have already taken the field theoretic matrix elements leaving (8) as an ordinary matrix element and we have specialized to $\pi^-$ capture.

If we choose the origin of our coordinate system in (8) as the center of the charge Ze that gives rise to $\varphi_\pi(\mathbf{r})$, then it is clear that the $\mathbf{v}_\pi$ term will be very nearly zero over the entire extent of the nucleus. Doing this we have

$$M_\zeta=\frac{G\mu\hbar N}{2mc^2}\frac{\varphi_\pi(0)}{L^{3/2}}\times\langle\chi^{s_z}e^{i\mathbf{k}_1\cdot\mathbf{r}_1}\psi_J{}^M(N-1)|\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}_1\tau_1^-|\psi(N)\rangle, \tag{9}$$

where the initial and final states are completely antisymmetric in all nucleons and $\zeta$ stands for the final quantum numbers which are the magnetic spin quantum

[14] We neglect this small Coulomb energy of the $\pi$.

[15] In Ref. 9 it is shown that this is indeed the capture orbit for capture on deuterons since the radiative transition $nl\to 1s$ dominates over capture from $nl$. As can be seen from (9) where $\varphi_\pi(0)$ appears, capture from $nl$ will rise as $Z^3Z^{2l}(r_0A^{1/3}/a_0)^{2l}$ while radiative transitions $nl\to ms$ go as $Z^6[(1/n^2)-(1/m^2)]^3(nm)^{3/2}Z^{-2}=Z^4[(n^2-m^2)^3/(nm)^{9/2}]$. Since $Z(r_0A^{1/3}/a_0)<1$ for all physical $Z$, the transition $nl\to ms$ is preferred over capture from $nl$ even more strongly as $Z$ increases. Similarly, it is easy to see that $nl\to ms$ is most rapid for $m=1$. Lastly, as the chances of getting caught in an $s$ orbit for a given $n$ is roughly $n^{-2}$ and since capture goes, from (9), as $n^{-3}$, we can see that $n=1$ dominates $n=2$ by 32 to 1 and so on. See, however, the discussion by G. A. Snow in the *Proceedings of the 1960 Annual International Conference on High Energy Physics at Rochester* (Interscience Publishers, Inc., New York, 1960), p. 407, on the mechanism of $K$-meson capture.

[16] K. Brueckner, Phys. Rev. 98, 769 (1955).

[17] F. Mandl, *Introduction to Quantum Field Theory* (Interscience Publishers, Inc., New York, 1959).

number $s_z$, the spin $J$ of the final nucleus, and its magnetic number $M$.

To evaluate (9) we assume that a given final state corresponds to a hole in a $j$ shell. Due to the high momentum of the outgoing particle we neglect the antisymmetry of this particle with the nucleons in the residual nucleus, and after some tedious algebra we can write

$$M_\zeta = AL^{-3/2}[2j+1]^{1/2} \times\langle \chi^{s_z} e^{i\mathbf{k}_1\cdot\mathbf{r}_1}\psi_J{}^M(2j)|\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}_1\tau_1{}^-|\psi_0(2j+1)\rangle, \quad (10)$$

with

$$A=(G\mu\hbar/2mc^2)\varphi_\pi(0)=1.60Z^{3/2}\times 10^{-21},$$

where we have taken[18]

$$G^2=16\pi(m/\mu)^2\hbar c f^2, \quad f^2=0.08.$$

The left-hand side of (10) is no longer antisymmetric in $\mathbf{r}_1$ and the wave functions $\psi_0(2j+1)$ and $\psi_J{}^M(2j)$ represent a closed shell and a closed shell less one particle, respectively.

Since

$$\psi_0(2j+1)=\sum_m \langle jjm-m|00\rangle \psi_j{}^m(2j)\psi_j{}^{-m}(\mathbf{r}_1),$$

we must have $J=j$, $M=+m$ and

$$\begin{aligned}\sum|M_\zeta|^2 &= \sum_{s_z,m} |\langle \chi^{s_z}e^{i\mathbf{k}\cdot\mathbf{r}}|\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}|\psi_j{}^{-m}(\mathbf{r})\rangle|^2\\ &=3\sum\sum \langle l\tfrac{1}{2}m-M-m|j-M\rangle\\ &\quad\times\langle l\tfrac{1}{2}m'-M-m'|j-M\rangle\\ &\quad\times\langle \tfrac{1}{2}1-M-m\; s_z+M+m|1s_z\rangle\\ &\quad\times\langle \tfrac{1}{2}1-M-m'\; s_z+M+m'|1s_z\rangle\\ &\quad\times\langle e^{i\mathbf{k}\cdot\mathbf{r}}|\nabla^{-m-M-s_z}|\psi_l{}^m(\mathbf{r})\rangle\\ &\qquad\times\langle e^{i\mathbf{k}\cdot\mathbf{r}}|\nabla^{-m'-M-s_z}|\psi_l{}^{m'}(\mathbf{r})\rangle. \quad (11)\end{aligned}$$

We use

$$e^{i\mathbf{k}\cdot\mathbf{r}}=4\pi\sum_{lm} i^l j_l(kr)Y_l{}^{m*}(\Omega_k)Y_l{}^m(\Omega_r)$$

and[19]

$$\begin{aligned}&\langle Y_{l'}{}^{m'}(\Omega_r)f(r)|\nabla^\mu|Y_l{}^m(\Omega_r)g(r)\rangle\\ &=(-1)^{m'}\begin{pmatrix} l' & 1 & l\\ -m' & \mu & m\end{pmatrix}\begin{pmatrix} l' & 1 & l\\ 0 & 0 & 0\end{pmatrix}^{-1}\\ &\qquad\times\langle Y_{l'}{}^0(\Omega_r)f(r)|\nabla^0|Y_l{}^0(\Omega_r)g(r)\rangle, \quad (12\text{a})\end{aligned}$$

where the matrix element on the right-hand side of (12a) is given by

$$\begin{aligned}&\langle f(r)Y_{l'}{}^0|\nabla^0|Y_l{}^0 g(r)\rangle\\ &=\frac{l+1}{[(2l+1)(2l+3)]^{1/2}}\left\langle f(r)\left|\frac{\partial}{\partial r}-\frac{l}{r}\right|g(r)\right\rangle, \quad l'=l+1 \quad (12\text{b})\\ &=\frac{l}{[(2l-1)(2l+1)]^{1/2}}\left\langle f(r)\left|\frac{\partial}{\partial r}+\frac{l+1}{r}\right|g(r)\right\rangle, \quad l'=l-1\\ &=0,\quad l'=l.\end{aligned}$$

Thus

$$\int d\Omega_k\sum|M_\zeta|^2=6(4\pi)^2(2j+1)\sum_L(2L+1) \times\langle 1l00|L0\rangle^{-2}\begin{Bmatrix} j & \tfrac{1}{2} & l\\ 1 & l & \tfrac{1}{2}\end{Bmatrix}^2|\langle L|\nabla^0|l\rangle|^2, \quad (13)$$

with $\langle L|\nabla^0|l\rangle$ defined by (12b) where $f(r)=j_L(kr)$ and $g(r)=\phi_l(r)$, the radial part of an $l$ shell-wave function which will be either 1$s$ or 1$p$ in $O^{16}$.

Making use of the general formula[20]

$$\left(\frac{2k}{\pi}\right)^{1/2}\int_0^\infty j_{n-\frac{1}{2}}(kr)r^{\mu-\frac{1}{2}}e^{-p^2r^2}dr=\frac{\Gamma(n/2+\mu/2)k^n}{2^{n+1}p^{n+\mu}\Gamma(n+1)} \times e^{-k^2/4p^2}{}_1F_1\left(\frac{n}{2}-\frac{\mu}{2}+1;n+1;\frac{k^2}{4p^2}\right), \quad (14)$$

where ${}_1F_1$ is the confluent hypergeometric function regular at the origin, we can readily evaluate (13). Doing so we have

$$\begin{aligned}\int d\Omega_k\sum|M_\zeta|^2&=396(k^2/\gamma^{3/2})e^{-k^2/2\gamma}, \quad l=0\\ &=440(k^4/\gamma^{5/2})e^{-k^2/2\gamma}, \quad l=1,\ j=\tfrac{3}{2}\\ &=132(k^4/\gamma^{5/2})e^{-k^2/2\gamma}, \quad l=1,\ j=\tfrac{1}{2},\end{aligned} \quad (15)$$

where our harmonic-oscillator wave functions are the usual ones and have an exponential form $\propto e^{-\gamma r^2}$. For purposes of numerical investigation we assume $p(\Delta)$ in (2) is very peaked at some $k$ so that it acts like a delta function and the integral over $p(\Delta)d\Delta$ gives unity. This is a good approximation for the $p$ shell in $O^{16}$ and somewhat less good for the $s$ shell. Then for $Z=8$,

$$\begin{aligned}T=\int N(W)dW&=18.8(k^3/\gamma^{3/2})e^{-k^2/2\gamma}\times 10^{18}, \quad l=0\\ &=20.9(k^5/\gamma^{5/2})e^{-k^2/2\gamma}\times 10^{18}, \quad l=1,\ j=\tfrac{3}{2}\\ &=6.29(k^5/\gamma^{5/2})e^{-k^2/2\gamma}\times 10^{18}, \quad l=1,\ j=\tfrac{1}{2}.\end{aligned} \quad (16)$$

[18] To make the dimensions of $A$ correct we have included implicitly an extra factor $(\hbar/\mu c)^2(\mu c^2)$ due to the fact that the field theoretic pion field defined in (5) has a different normalization than the wave function used in (8).

[19] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1960), 2nd ed., p. 80.

[20] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, New York, 1944), 2nd ed., Chap. XIII.

Finally, we take[21] $\gamma=0.178\ f^{-2}$ and for the value of $k$ we have

$$k=2.19\ f^{-1}\quad (40\text{ MeV})\quad s_{1/2}$$
$$k=2.42\ f^{-1}\quad (18\text{ MeV})\quad p_{3/2}$$
$$k=2.49\ f^{-1}\quad (11.13\text{ MeV})\quad p_{1/2}.$$

The energy in brackets after each $k$ gives the corresponding separation energy assumed for a proton in the specified state. These values of energy are taken from Pugh and Riley.[22]

We have

$$\begin{aligned} T&=3.65\times10^{15}\text{ sec}^{-1}\quad s_{1/2},\\ T&=9.31\times10^{15}\text{ sec}^{-1}\quad p_{3/2},\\ T&=1.23\times10^{15}\text{ sec}^{-1}\quad p_{1/2},\\ T_{\text{Total}}&=1.42\times10^{16}\text{ sec}^{-1}\end{aligned}\tag{17}$$

and we see that the $k$ dependence has reduced $T$ by several orders of magnitude from the maximum value obtainable in (16). This is not surprising since we are seeking out the high-momentum components of the shell-model wave functions and these are small.

## III. EFFECT OF A DISTORTED WAVE

If we assume a complex nuclear well of the form

$$V(r)=-iV_0e^{-\beta r^2},\tag{18}$$

then Squires[23] has shown that an outgoing distorted wave of momentum $\mathbf{k}$ and energy $E$ can be written as

$$\chi^-=e^{i\mathbf{k}\cdot\mathbf{r}+iS},\qquad S=\frac{ikV_0}{2E}\int_{(x,y,z)}^{\infty}e^{-\beta r^2}dt,\tag{19}$$

where the path of integration is along $\mathbf{k}$. This can be written as

$$S=\frac{ikV_0}{2E}e^{-\beta r^2\sin^2\varphi}\int_{r|\cos\varphi|}^{\infty}e^{-\beta t^2}dt\tag{20}$$

with $\cos\varphi=\mathbf{k}\cdot\mathbf{r}/kr$. It should be noted that $\theta=\varphi$ ($\theta$ is the azimuthal angle of $\mathbf{r}$) *only* when $\mathbf{k}$ is along the $z$ axis.

If we expand $\chi^-$ in a series of spherical harmonics as was done with $e^{i\mathbf{k}\cdot\mathbf{r}}$ we will have coefficients that are not simply spherical Bessel functions but rather integrals like

$$h_l(r)=\int\int d\Omega_r d\Omega_k e^{i\mathbf{k}\cdot\mathbf{r}+iS}Y_l^{m*}(\Omega_r)Y_l^m(\Omega_k).\tag{21}$$

Since (21) is a double integral over the angles of $\mathbf{k}$ and $\mathbf{r}$, we will replace, in $S$, $\sin^2\varphi$ by $\frac{2}{3}$ and $|\cos\varphi|$ by $\frac{1}{2}$, which are the average values of these functions over a sphere. Then instead of $j_l(kr)$ we have

$$h_l(kr)=4\pi i^l j_l(kr)e^{iS'},\qquad S'=\frac{ikV_0}{2E}e^{-\frac{2}{3}\beta r^2}\int_{\frac{1}{2}r}^{\infty}e^{-\beta t^2}dt.\tag{22}$$

We can now use Eq. (13) with $j_L(kr)e^{iS'}$ in place of $j_L(kr)$ and instead of using (14) we have done a machine integration on the IBM 1620 at the University of Rochester. With $V_0=60$ MeV and[24] $\beta=0.215\ f^{-2}$ and for the same values of $k$ as before we have[25]

$$\begin{aligned} T&=2.74\times10^{16}\text{ sec}^{-1},\\ T&=6.65\times10^{16}\text{ sec}^{-1},\\ T&=4.64\times10^{16}\text{ sec}^{-1},\\ T_{\text{Total}}&=1.40\times10^{17}\text{ sec}^{-1}.\end{aligned}\tag{23}$$

These values are significantly enhanced over the values for a plane wave given in (17).

## IV. TWO-NUCLEON EJECTION

The consideration of the question of two-nucleon ejection raises the problems of two-body correlations in the initial state and of the nucleon-nucleon interaction of the ejected pair in the final state. The radial matrix elements that arise can be expressed in terms of center-of-mass and relative coordinates, and it is in the latter that a correlation function should be used. In many respects our calculation is similar to one of Reitan[26] on quasideuteron production in $O^{16}$ by 100-MeV gamma rays. He refers there to the work of DeSwart and Marshak[27] which indicates that the relative coordinate integrals are not sensitive to the small $r$ dependence of the wave functions for momenta in the region corresponding to 100 MeV. This small $r$ region is just that region where a correlation function differs most from unity. Explicit calculations by Reitan[28] using no correlation function give results very similar to the numbers, obtained with a correlation function, in his published paper.

It has been suggested by Brown[28] that the same conclusions are applicable to the pion capture process since the argument against the importance of small $r$ behavior is due mainly to questions of momentum

[21] S. Iwao, thesis, University of Rochester, 1960 (unpublished).

[22] H. G. Pugh and K. F. Riley, in *Proceedings of the Rutherford Jubilee International Conference* (Heywood and Company, Ltd., London, 1960), p. 195.

[23] E. J. Squires, Nucl. Phys. **6**, 504 (1958).

[24] We determine $\beta$ for $O^{16}$ the same way Squires (Ref. 23) determined $\beta$ for $C^{12}$: By making the rms of the potential correspond to the Hofstadter value; see R. Hofstadter, Ann. Rev. Nucl. Sci. **7**, 231 (1957).

[25] If we calculate a value for $T$ for $He^4$ using an average enhancement of 10 due to the distorted waves as determined by a comparison of (17) and (23), we find $T(He^4+\pi^-\rightarrow H^3+n)\approx10^{16}\text{ sec}^{-1}$ which is an order of magnitude less than Eckstein's value. Of course, there is no reason to expect that we should obtain agreement as the two methods of calculation are not equivalent.

[26] Arne Reitan, Nucl. Phys. **36**, 56 (1962).

[27] J. J. deSwart and R. E. Marshak, Phys. Rev. **111**, 272 (1958).

[28] Gerry Brown (private communication).

transfer. A momentum transfer corresponding to[29] 50 MeV gives roughly a Heisenberg uncertainty distance of 0.90 $f$ whereas the Compton wavelength of the nucleon is 0.21 $f$.

We are inclined to accept this argument by Brown since our integrals are very similar to those of Reitan, and in that case there is numerical evidence in favor of this argument. As a result the ensuing integrals are much simplified. We leave until later the question of the final-state scattering interaction.

In the two-nucleon case, (2) is replaced by

$$N'(W,\Omega_1,\Omega_2)dWd\Omega_1 d\Omega_2 = (2\pi/\hbar)|M|^2\rho(k_1)\rho(k_2)dE_1dE_2 \times p(\Delta)d\Delta\delta[\mu c^2-\Delta-(\hbar^2/2m)(k_1^2+k_2^2)], \quad (24)$$

or if we define $\mathbf{k}=\frac{1}{2}(\mathbf{k}_1-\mathbf{k}_2)$, $\mathbf{K}=\mathbf{k}_1+\mathbf{k}_2$,

$$N(W)dW=\frac{L^6m^2}{(2\pi\hbar)^5}kKp(\Delta)d\Delta dE_K \times\int\int d\Omega_k d\Omega_K|M|^2. \quad (25)$$

In (25), $\Delta$ represents the separation energy for two particles, but $p(\Delta)d\Delta$ has the same meaning as before. We now have, with the wave function of the ejected particles given by $\psi(\mathbf{r}_1,\mathbf{r}_2)$,

$$M_\zeta=AN\langle\psi(\mathbf{r}_1,\mathbf{r}_2)\psi(N-2)|\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}_1\tau_1^-|\psi(N)\rangle, \quad (26)$$

where initial and final states are completely antisymmetric in all nucleons. If we assume that a definite final state corresponds to two $j$ holes coupled to some $J$ and some $M_J$ we can show with some tedious algebra that, again neglecting exchange between the ejected pair and the final bound nucleons,

$$M_\zeta=A\sigma_{j_1j_2}{}^J\langle\{\psi(1,2)\chi_S{}^{M_S}(1,2)\chi_T{}^{M_T}(1,2)\} \times\psi_{JM_J}{}^{T'M_{T'}}(N-2)|\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}_1\tau_1^-|\psi(N)\rangle, \quad (27)$$

where $\psi(1,2)$ represents the spatial wave function of the ejected pair, $\chi_S{}^{M_S}$ and $\chi_T{}^{M_T}$ are the spin and isotopic spin functions of the pair, and $\psi_{JM_J}{}^{T'M_{T'}}$ is the wave function for the $(N-2)$ particle daughter nucleus with indicated quantum numbers. In (27), $\sigma_{j_1j_2}{}^J=[(2j_1+1)\times(2j_2+1)]^{1/2}$ if $j_1\neq j_2$ and $\sigma_{j_1j_2}{}^J=[j(2j+1)]^{1/2}$ if $j_1=j_2$, both protons. In the case of one proton and one neutron, we have $2\sigma_{j_1j_2}{}^J$, $j_1\neq j_2$ and $j_1=j_2$ for even $J$. For odd $J$ and $j_1=j_2$ we have $2[(j+1)/j]^{1/2}\sigma_{j_1j_2}{}^J$.

We take the complete two-particle wave function on the left side of (27) to be antisymmetric in the pair but now not antisymmetrized with $\psi(N-2)$. We wish to know the value of $|M_\zeta|^2$ summed over all final states and integrated over all angles of $\mathbf{K}$ and $\mathbf{k}$. This is given by

$$\mathfrak{M}^2=\int\int d\Omega_k d\Omega_K\sum|M_\zeta|^2 =\sum_{S,M_S}\sum_{T,M_T}\sum_{J,M_J}\sum_{T',M_{T'}} \times A^2(\sigma_{j_1j_2}{}^J)^2\int\int d\Omega_k d\Omega_K|\langle\{\psi\chi_S{}^{M_S}\chi_T{}^{M_T}\} \times\psi_{JM_J}{}^{T'M_{T'}}|\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}_1\tau_1^-|\psi(N)\rangle|^2, \quad (28)$$

with

$$\psi(N)=\sum_{J''T''}\alpha^{J''T''}\sum_{M_{J''}M_{T''}}\langle J''J''M_J''-M_J''|00\rangle \times\langle T''T''M_T''-M_T''|00\rangle\psi_{J''}{}^{M_J''}(1,2) \times\chi_{T''}{}^{M_T''}(1,2)\psi_{J''T''}{}^{-M_J''-M_T''}(1,2),$$

where $\psi_{J''}{}^{M_J''}(1,2)\chi_{T''}{}^{M_T''}(1,2)$ is antisymmetric in particles one and two. Here $\alpha^{JT}$ is given by, with $j_1$ and $j_2$ the initial $j$ states of the two nucleons:

$$(\alpha^{JT})^2=\frac{2J+1}{j(j+1)}, \quad \text{even } J,\ j_1=j_2, \text{ two protons},$$
$$=0, \quad \text{odd } J,\ j_1=j_2, \text{ two protons},$$
$$=\frac{1}{4}\left[\frac{2J+1}{j(j+1)}\right], \quad \text{even } J,\ j_1=j_2, \text{ neutron-proton}$$
$$=\frac{3}{4}\left[\frac{2J+1}{(j+1)(2j+1)}\right], \quad \text{odd } J,\ j_1=j_2, \text{ neutron-proton}$$
$$=\frac{2J+1}{(2j_1+1)(2j_2+1)}, \quad \text{all } J,\ j_1\neq j_2, \text{ two protons}$$
$$=\frac{2J+1}{(2j_1+1)(2j_2+1)}\frac{2T+1}{4}, \quad \text{all } J,\ T,\ j_1\neq j_2, \text{ neutron-proton}.$$

We now have

$$\mathfrak{M}^2=4A^2\sum|\langle\chi_T{}^{M_T}|\tau_1^-|\chi_{T'}{}^{-M_{T'}}\rangle|^2\int\int d\Omega_k d\Omega_K \times|\langle\psi(1,2)\chi_S{}^{M_S}|\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}_1|\psi_J{}^{-M_J}(1,2)\rangle|^2, \quad (29)$$

where $\psi_J{}^{-M_J}(1,2)$ is symmetric or antisymmetric depending on $\chi_{T'}{}^{-M_{T'}}$. The 4 in (29) comes from the fact that we no longer antisymmetrize the final state. As

$$\langle\chi_T{}^{M_T}|\tau_1^-|\chi_{T'}{}^{-M_{T'}}\rangle =(-1)^{M_{T'}}\sqrt{2}[(2T+1)(2T'+1)]^{1/2} \times\langle TT'M_T-M_{T'}|1-1\rangle\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & T\\ 1 & T' & \frac{1}{2}\end{Bmatrix}$$

[29] This is approximately the maximum momentum transfer available in the two-nucleon ejection process.

and since we can sum over the indices $T$, $M_T$ and $M_T$ in (29) we have

$$\mathfrak{M}^2=4A^2\sum_{S,M_S}\sum_{J,M_J}\sum_{T'}(2T'+1)\int\int d\Omega_k d\Omega_K \times|\langle\psi(\mathbf{r},\mathbf{R})\chi_S{}^{M_S}|\boldsymbol{\sigma}_1\cdot(\boldsymbol{\nabla}_r+\tfrac{1}{2}\boldsymbol{\nabla}_R)|\psi_J{}^{-M_J}(\mathbf{r}_1,\mathbf{r}_2)\rangle|^2, \quad (30)$$

where $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ and $\mathbf{R}=\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)$.

Using

$$\psi_J{}^{-M_J}(\mathbf{r}_1,\mathbf{r}_2)=\sum_{LS'}\sum_{M_L}\langle LS'M_LM_S'|J-M_J\rangle \times[(2L+1)(2S'+1)(2j_1+1)(2j_2+1)]^{1/2} \times\begin{Bmatrix} l_1 & l_2 & L\\ \frac{1}{2} & \frac{1}{2} & S\\ j_1 & j_2 & J\end{Bmatrix}\psi_L{}^{M_L}(\mathbf{r}_1,\mathbf{r}_2)\chi_{S'}{}^{M_S'}(1,2)$$

we can evaluate the spin matrix element in (30) yielding

$$\mathfrak{M}^2=8A^2\sum_{S,M_S}\sum_{J,M_J}\sum_{T'}(2T'+1)(2S+1) \times\int\int d\Omega_k d\Omega_K\Bigg|\sum_{L,M_L}\sum_{S',M_S'}(2L+1)^{1/2}(2S'+1) \times\langle LS'M_LM_S'|J-M_J\rangle \times\langle SS'M_S-M_S'|1M_S-M_S'\rangle \times\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & S\\ 1 & S' & \frac{1}{2}\end{Bmatrix}\begin{Bmatrix} l_1 & l_2 & L\\ \frac{1}{2} & \frac{1}{2} & S'\\ j_1 & j_2 & J\end{Bmatrix} \times f(-M_S+M_S',M_L,L)\Bigg|^2 \quad (31)$$

with

$$f(-M_S+M_S',M_L,L) =\langle\psi(\mathbf{r},\mathbf{R})|(\nabla_r+\tfrac{1}{2}\nabla_R)^{-M_S+M_S'}|\psi_L{}^{M_L}(\mathbf{r}_1,\mathbf{r}_2)\rangle$$

where again the symmetry properties of $\psi_L{}^{M_L}(\mathbf{r}_1,\mathbf{r}_2)$ depend on those of $\chi_{S'}{}^{M's}$ and $\psi_J{}^{-M_J}$.

We can convert $\psi_L{}^{M_L}(\mathbf{r}_1,\mathbf{r}_2)$ to a function of $\mathbf{r}$ and $\mathbf{R}$ to get

$$f(-M_S+M_S',M_L,L) =\sum_{m_l}\sum_{l',l'',m_l'}\langle l_1l_2m_lM_L-m_l|LM_L\rangle \times A_{l'l''}{}^{l_1l_2}(m_l',M_L-m_l') \times\langle\psi(\mathbf{r},\mathbf{R})|(\nabla_r+\tfrac{1}{2}\nabla_R)^{-M_S+M_S'}| \times\psi_{l'}{}^{m_l'}(\mathbf{r})\psi_{l''}{}^{M_L-m_l'}(\mathbf{R})\rangle, \quad (32)$$

where the coefficients $A_{l'l''}{}^{l_1l_2}(m_l',M_L-m_l')$ are given by Talmi.[30] We note the important fact that if two-harmonic oscillator radial wave functions have an exponential dependence $\varphi(r_1)\propto e^{-\gamma r_1^2}$ and similarly for $r_2$ then we will have $\varphi(r)\propto e^{-\frac{1}{2}\gamma r^2}$ and $\varphi(R)\propto e^{-2\gamma R^2}$.

We take $e^{i\mathbf{K}\cdot\mathbf{R}}$ for the $\mathbf{R}$ coordinate wave function, as does Reitan, but for the relative coordinate the problem is more difficult. DeSwart and Marshak[27] have shown that the asymptotic radial form

$$\phi_l(r)=4\pi i^l e^{i\delta_l}[\cos\delta_l j_l(kr)-\sin\delta_l\eta_l(kr)] \quad (33)$$

will give good results in such integrals as ours for energies around 100 MeV. It is not always necessary to truncate the Neumann function since there may be enough powers of $r$ at the origin in the integral to compensate for the ill behavior of $\eta_l(kr)\sim r^{-l-1}$. This is the case in our problem. If we want the effect of a plane wave we take

$$\phi_l(r)=4\pi i^l j_l(kr).$$

We record here the matrix elements we will need as

$$\langle e^{i\mathbf{K}\cdot\mathbf{R}}|\psi_{l''}{}^0(\mathbf{R})\rangle=(4\pi)i^{l''}(-1)^{l''}Y_{l''}{}^0(\Omega_K)\times\int_0^\infty R^2dRj_{l''}(KR)\varphi_{l''}(R)$$

$$\langle e^{i\mathbf{K}\cdot\mathbf{R}}|\nabla_R{}^0|\psi_{l''}{}^0(\mathbf{R})\rangle=i\left(\frac{4\pi}{3}\right)^{1/2}KY_1{}^0(\Omega_K)\times\langle e^{i\mathbf{K}\cdot\mathbf{R}}|\psi_{l''}{}^0(\mathbf{R})\rangle, \quad (34)$$

$$\langle\psi(\mathbf{r})|\psi_{l'}{}^0(\mathbf{r})\rangle=Y_{l'}{}^0(\Omega_k)\int_0^\infty r^3dr\phi_{l'}{}^*(r)\varphi_{l'}(r),$$

$$\langle\psi(\mathbf{r})|\nabla_r{}^0|\psi_{l'}{}^0(\mathbf{r})\rangle=\langle a_{l'-1}\psi(\mathbf{r})|\psi_{l'-1}{}^0(\mathbf{r})\rangle+\langle a_{l'+1}\psi(\mathbf{r})|\psi_{l'+1}{}^0(\mathbf{r})\rangle,$$

where $a_{l'-1}$ and $a_{l'+1}$ are coefficients which are easily determined for any specific case.

## V. EVALUATION OF EQ. (31)

Due to the coefficients $A_{l'l''}{}^{ll}(m_l',M_L-m_l')$ in (32), it is not possible to form a closed expression for $\sum|M_\zeta|^2$, and the enormous effort of evaluating so many terms individually is not sufficiently rewarding to justify the effort since we are able to determine several features of the capture by making some suitable approximations. As the radial integrals determine the magnitude as well as other features of the process, we make the approximation that

$$\sum|M_\zeta|^2=|\sum_{l'l''}\langle\psi(\mathbf{r},\mathbf{R})|(\nabla_r+\tfrac{1}{2}\nabla_R)^0| \times\psi_{l'}{}^0(\mathbf{r})\psi_{l''}{}^0(\mathbf{R})\rangle|^2. \quad (35)$$

Dropping the magnetic quantum number dependence of the integrals in (31) will obviously not affect either

[30] I. Talmi, Helv. Phys. Acta **25**, 185 (1952).

the order of magnitude nor the angular features (see below) of these integrals. Also, since the value of all the quantum numbers in the various Clebsch-Gordan coefficients, 6-$j$ symbols, and 9-$j$ symbols in (31) is small, the approximation of setting each numerical coefficient equal to $1/m$ in an $m$ term sum (giving a result of unity) will not affect the final answer very much.

We now investigate the maximum value of $\mathfrak{M}^2$ for the case of back-to-back ($b$-$b$) ejection of two particles ($\mathbf{K}=0$) and show that in general this will be larger than parallel ($p$) ejection ($\mathbf{k}=0$).

In order to see what term of the type (35) gives the largest effect we need to know that $\eta_l(kr)=(-1)^{l+1}\times j_{-l}(kr)$ and that an integral over $r^{l+2}n_l(kr)e^{-\gamma r^2/2}$ is proportional to $k^{-l-2}$ for large $k$. This is immediate from (14) and the relation

$$ {}_1F_1(a,c,z)\rightarrow[\Gamma(c)/\Gamma(a)]e^z z^{a-c}+O(z^{a-c-1}) \quad \text{as} \quad z\rightarrow+\infty\,. $$

However integrals containing $j_l(kr)$ result in a factor $e^{-k^2/2\gamma}$ and obviously the largest possible terms will come from the smallest possible $k$. This would be the ejection of one $s$-shell proton and one $s$-shell neutron where the final kinetic energy would be 54.7 MeV or $k=1.15\ f^{-1}$ since $E_k=\hbar^2k^2/m$ for relative momentum (and $E_K=\hbar^2K^2/4m$ for total momentum).

With a little computation it is possible to establish the following facts: For any $k$ obtainable in this problem for back-to-back ejection, *except* the smallest one given above, the integral over $j_l(kr)$ is smaller than that over $\eta_l(kr)$ for those $l$'s that appear. For this smallest $k$, however, we find that the integral over $j_0(kr)$ is larger than that over $\eta_0(kr)$ which, in turn, is larger than $\eta_l(kr)$ for any other $l$ and $k$ arising. This leads us to the following dominate expression

$$ \sum|M_\zeta|^2=|\langle e^{i\mathbf{K}\cdot\mathbf{R}}|\psi_0{}^0(\mathbf{R})\rangle\langle\psi(\mathbf{r})|\nabla_r{}^0|\psi_0{}^0(\mathbf{r})\rangle|^2\,, \tag{36} $$

where we retain only the largest term for $\mathbf{K}\approx0$. Now we have

$$ |\langle e^{i\mathbf{K}\cdot\mathbf{R}}|\psi_0{}^0(\mathbf{R})\rangle|^2=(4\pi)\left|\int_0^\infty R^2dRj_0(KR)\varphi_0(R)\right|^2 = \pi^{3/2}\gamma^{-3/2}e^{-K^2/4\gamma}\,, \tag{37a} $$

while

$$ \int\int d\Omega_k d\Omega_K|\langle\psi(\mathbf{r})|\nabla_r{}^0|\psi_0{}^0(\mathbf{r})|^2=\frac{256}{3}\pi^{7/2}\gamma^{7/2}A^2(k)\,, $$

$$ \begin{aligned} A^2(k)&=\left|\int_0^\infty r^3dre^{-\gamma r^2/2}[\cos\delta_1 j_1(kr)-\sin\delta_1\eta_1(kr)]\right|^2 \\ &=\left|\left(\frac{\pi}{2}\right)^{1/2}\frac{k}{\gamma^{5/2}}e^{-k^2/2\gamma}\cos\delta_1-\frac{3}{k^4}\sin\delta_1\right|^2 \\ &\approx\frac{\pi}{2}\frac{k^2}{\gamma^5}e^{-k^2/\gamma}\,, \end{aligned} \tag{37b} $$

where we put $\cos\delta_1=1$ for a maximum effect. Combining (37) with (25) we have for $\mathbf{K}\approx0$,

$$ \begin{aligned} N(E_K)dE_K&=\frac{A^2m^2}{(2\pi\hbar)^5}kK\left(\frac{128\pi^6}{3}\right)\gamma^{-3}k^2e^{-k^2/\gamma-K^2/4\gamma}dE_K \\ &=12\frac{k^3K}{\gamma^3}e^{-k^2/\gamma-K^2/4\gamma}dE_K\times10^{48}\,, \end{aligned} $$

where $k=\frac{1}{2}[4m\Delta/\hbar^2-K^2]^{1/2}$ with $\Delta=54.7$ MeV. We then have as the total transition rate

$$ \begin{aligned} T&=\int_0^\Delta N(E_K)dE_K \\ &=(7.76\times10^{18})\int_0^1[1-S]^{3/2}S^{1/2}e^{-12.9S}ds\,, \\ S&=\frac{\hbar^2K^2}{4m\Delta}\,. \end{aligned} \tag{38} $$

This expression for $T$ is actually incorrect since the integrand appearing in (38) comes from an analytic expression valid only near $\mathbf{K}=0$ as this was the only range in which (33) is considered valid. We can see that the integrand in (38) peaks near $S=0$ ($\mathbf{K}\approx0$) which means that back-to-back ejection is the dominate contribution in this region. If we can show that the value of $\sum|M_\zeta|^2$ for parallel ejection ($\mathbf{k}=0$) will not exceed its value for $\mathbf{K}=0$, then we may use (38) to obtain an estimate of the total transition rate for two-nucleon ejection.

To investigate the question of parallel ejection ($\mathbf{k}=0$), it is clear after a little thought that the same expressions will appear as we had in (37) and (38) if we replace $\nabla_r$ by $\frac{1}{2}\nabla_R$ which will reverse the values of $\mathbf{k}$ and $\mathbf{K}$. This is permissible only if the expression (33) can be used for $k\approx0$ since we have only shown that it is good for large $k$. We show that (33) is acceptable for $k\approx0$ as well by considering the nucleon-nucleon potential to be a square well.

Consider a square well of depth $V_0=50$ MeV and range $r_0=\hbar/mc$ which we use to represent the nucleon-nucleon interaction. If the energy of the system is very small $V_0\gg E>0$, then the wave functions for the system are

$$ \begin{aligned} \psi_{\rm IN}&=j_l(\alpha r)\,, \\ \psi_{\rm OUT}&=A_l(k)[j_l(kr)-\tan\delta_l(k)\eta_l(kr)]\,, \\ \alpha^2&=\frac{\hbar^2}{2m}(E+V_0)\approx\frac{\hbar^2V_0}{2m}\,, \\ \tan\delta_l(k)&\approx\delta_l(k)\,,\quad A_l(k)\approx1\,. \end{aligned} \tag{39} $$

We consider the integral, where $\varphi(r)$ is a bound-harmonic oscillator wave function,

$$I(k)=\int_0^{r_0} j_l(\alpha r)\varphi(r)r^2dr+\int_{r_0}^{\infty}[j_l(kr)-\delta_l(k)\eta_l(kr)]\varphi_l(r)r^2dr. \quad (40)$$

Since $\alpha r_0\gg 1$ we can neglect the first integral, as the oscillation of $j_l(\alpha r)$ will cause it to be small, and since $kr_0\ll 1$ we can replace the lower limit in the second integral in (40) by zero then

$$I(k)\propto\left[k^l-ck^m\int_0^{\infty}\eta_l(kr)\varphi_l(r)r^2dr\right],$$

$$I(k)\propto[k^l-ck^{m-l-1}]\quad(m-l-1)\geqslant 0,\quad c=\text{constant}.$$

The condition $(m-l-1)\geqslant 0$ is due to the fact that $\delta_l(k)\eta_l(kr)$ must approach a constant or zero as $k\rightarrow 0$. We see that we can hope at best for a contribution from the $r$ integral that is of order unity [compared to $(k/\gamma^{1/2})e^{-k^2/2\gamma}$ for example in (37b)] when $k$ is small.

Thus we see that for two $s$-shell particles the parallel ejection will be of the order of or less than back-to-back ejection. The reason for this is that for such as low value of $k$, the plane-wave term in (33) is actually more important than the term due to the nucleon-nucleon potential when it comes to back-to-back ejection. Hence, parallel ejection will be about the same. However, this will not be true for ejection of other pairs of particles since the interaction term $\eta_l(kr)$ will then be more important than the plane-wave term. It is possible with some numerical work to see that, *except* for the one case above, back-to-back ejection will be larger than parallel ejection by an order of magnitude or more.

We can see from all the above that the transition rate as a function of angle between the two particles will fall on going from 180 to 0°, but it will not do so monotonically as there are terms in (35) like $K^m e^{-K^2/4\gamma}$ which do not peak at $\mathbf{K}=0$. As the leading parallel term is about the size of the leading back-to-back term, the total fall of the transition rate with angle should not be excessive and probably not exceed a factor of 10. Experiment[5] indicates a fall of about a factor of 5 from 180 to 90°.

Evaluating (38) by putting $[1-S]^{3/2}S^{1/2}=1$ we have

$$T_{\text{Total}}{}^{\text{Max}}=6.0\times 10^{17}\ \text{sec}^{-1}.$$

This is to be compared with (23).

## VII. CONCLUSION

We enumerate the points of interest in this paper:

(1) Single-nucleon ejection is not strongly suppressed compared to two-nucleon ejection if the effects of a distorted wave are included.

(2) The transition rate as a function of angle between two ejected particles will not monotonically decrease from 180 to 0° but will have structure.

(3) In general, low-lying $s$ particles will enter into the capture reaction dominantly and capture is not limited to surface effects.

We have not yet mentioned the question of the relative production of two neutrons compared to one proton and one neutron. Experimentally[5] this ratio is about 5. We see that Eq. (29) in principle will determine a value for this ratio by not summing over the initial and final isotopic magnetic quantum numbers of the pair. Of course an exact evaluation of all the terms in (29) would be necessary but would probably still not give the correct answer. This is because we have neglected Coulomb forces as well as the effect of triplet versus singlet forces. Both these effects will tend to let a neutron and a proton be more closely correlated in space than two protons. This question has been dealt with elsewhere.[31]

*Note added in proof.* H. L. Anderson, E. P. Hincks, C. S. Johnson, C. Rey, and A. M. Segar, Phys. Rev. **133**, B392 (1964), give experimental results for high-energy neutron ejections (their Table III) which tend to support the value of the ratio of our numbers for one- and two- nucleon ejection.

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[31] P. Huguenin, Nucl. Phys. **41**, 534 (1963).